Application of Laplace's Equation to Spherical Symmetry

Introduction

Laplace's equation is a second-order partial differential equation that plays a crucial role in various fields such as electrostatics, fluid dynamics, and gravitational theory. It is given by:

$$\nabla^2\phi = 0$$

where ϕ is a scalar potential function. In spherical coordinates (r, θ, ϕ), Laplace's equation is particularly useful for problems exhibiting spherical symmetry.

Spherical Coordinates

In spherical coordinates, the Laplacian operator is expressed as:

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\phi^2}$$

For problems with spherical symmetry, the potential ϕ depends only on the radial coordinate r, simplifying the equation to:

$$\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 0$$

Solution to Laplace's Equation in Spherical Symmetry

To solve this, we integrate the equation twice. First, we integrate with respect to r:

$$\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 0$$

Integrating once gives:

$$r^2\frac{d\phi}{dr} = C_1$$

where C_1 is a constant of integration. Integrating again with respect to r:

$$\frac{d\phi}{dr} = \frac{C_1}{r^2}$$

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$$\frac{d\phi}{dr} = \frac{C_1}{r^2}$$

$$\phi(r) = -\frac{C_1}{r} + C_2$$

where C_2 is another constant of integration. Hence, the general solution to Laplace's equation in spherical symmetry is:

$$\phi(r) = \frac{A}{r} + B$$

where $A = -C_1$ and $B = C_2$.

Physical Interpretation and Boundary Conditions

The constants A and B are determined by the boundary conditions of the specific physical problem. For instance, in electrostatics, ϕ might represent the electric potential due to a point charge. If the potential approaches zero at infinity, B is zero. If there is a charge Q at the origin, $\phi(r) = \frac{Q}{4\pi\epsilon_0 r}$.

Example: Electrostatic Potential

Consider the electrostatic potential outside a spherical charge distribution. The boundary condition might be that the potential at the surface of the sphere of radius R is $\phi(R) = \phi_0$. Applying this boundary condition:

$$\phi(R) = \frac{A}{R} + B = \phi_0$$

If we also know that the potential must vanish at infinity ($\phi(\infty) = 0$), then $B = 0$. Thus,

$$\phi(r) = \frac{A}{r}$$

Solving for A using the boundary condition at $r = R$,

$$\phi_0 = \frac{A}{R}$$

$$A = \phi_0 R$$

Therefore, the potential is:

$\phi(r) = \phi_0 R/r$

Conclusion

Laplace's equation in spherical symmetry simplifies significantly and is pivotal in solving many physical problems, particularly in electrostatics. By leveraging the symmetry of the problem, we reduce a complex partial differential equation to a more manageable form, yielding insights into the behavior of physical systems.