Application of Laplace's Equation to Planar Symmetry

Laplace's equation is a second-order partial differential equation widely used in physics and engineering. It describes the behaviour of scalar fields like electrostatic potential, fluid velocity potential and temperature distribution. The equation is given by:

$$\nabla^2\phi=0$$

Where, ∇^2 is the Laplacian operator and ϕ is the scalar field.

Planar Symmetry

Planar symmetry implies that the problem can be described in a two-dimensional plane. In Cartesian coordinates, for a scalar function $(x,)$, Laplace's equation in two dimensions is:

$$\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2=0$$

Example: Electrostatic Potential in a Rectangular Region

Consider a rectangular region with dimensions $a\times b$. The electrostatic potential $(x,)$ in this region is governed by Laplace's equation:

$$\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2=0$$

Boundary Conditions:

To solve this equation, we need boundary conditions. Suppose the potential is specified on the boundaries of the rectangle:

1. $\phi(0,y)=f^1(y)$
2. $\phi(a,y)=f^2(y)$
3. $\phi(x,0)=g^1(x)$
4. $\phi(x,b)=g^2(x)$

Solution Method:

One common method to solve this is separation of variables, where we assume:

$$\phi(x,y)=X(x)Y(y)$$

Substituting into Laplace's equation:

$$X''(x)Y(y)+X(x)Y''(y)=0$$

Dividing through by $(x)(y)$

$$X''(x)/X(x)+Y''(y)/Y(y)=0$$

Since the left side depends only on x and the right side only on y, both must be equal to a separation constant, say $-\lambda$:

$$X''(x)/X(x)=-\lambda \text{ and } Y''(y)/Y(y)=\lambda$$

This gives us two ordinary differential equations:

$$X''(x)+\lambda X(x)=0 \text{ and } Y''(y)-\lambda Y(y)=0$$

The general solutions to these equations are:

$$X(x)=A\cos(\sqrt{\lambda}_x)+B\sin(\sqrt{\lambda}_x)$$

$$Y(y)=C\,e^{\lambda y}+De^{-\lambda y}$$

The constants A, B, C, and D are determined by the boundary conditions.

Diagram:

Here is a diagram representing the rectangular region with boundary conditions:

```
Y
^
|
| g^2(x)
|
|
+-------------------> x
|
|
|
|
| g^1(x)
|
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In this diagram, $g^1(x)$ and $g^2(x)$ represent the potential on the horizontal boundaries (at y=0 and y=b), while $f^1(y)$ and $f^2(y)$ represent the potential on the vertical boundaries (at x=0 and x=a).

[NB: Application in Physics

Electrostatics: Determining the electric potential in a region with fixed potentials on the boundaries.

Heat Conduction: Finding the steady-state temperature distribution in a flat plate with specified temperatures on the edges.

Fluid Dynamics: Describing the potential flow in two-dimensional incompressible fluid flow problems.

By solving Laplace's equation with the appropriate boundary conditions, one can find the scalar field $\phi(x,y)$ that satisfies the physical constraints of the problem.]

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