Date 30/09/12

Electric field due to a uniformly charged wire (Rod):

i) Let us consider a uniformly charged rod of linear charge density λ. Let us find electric field

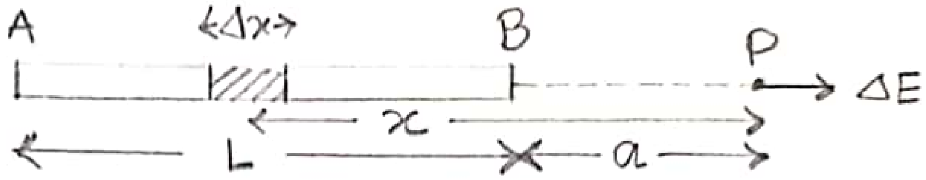


Fig. 1

at axial point P due to this rod.

The electric field at point P due to length element Δx is

$$\Delta E = \frac{1}{4\pi\epsilon_0}\frac{\lambda \Delta x}{x^2} \qquad \left[\because \lambda = \frac{q}{L}\right]$$

The direction of ΔE is along BP, since λ is +ve charge density.

The net field will be

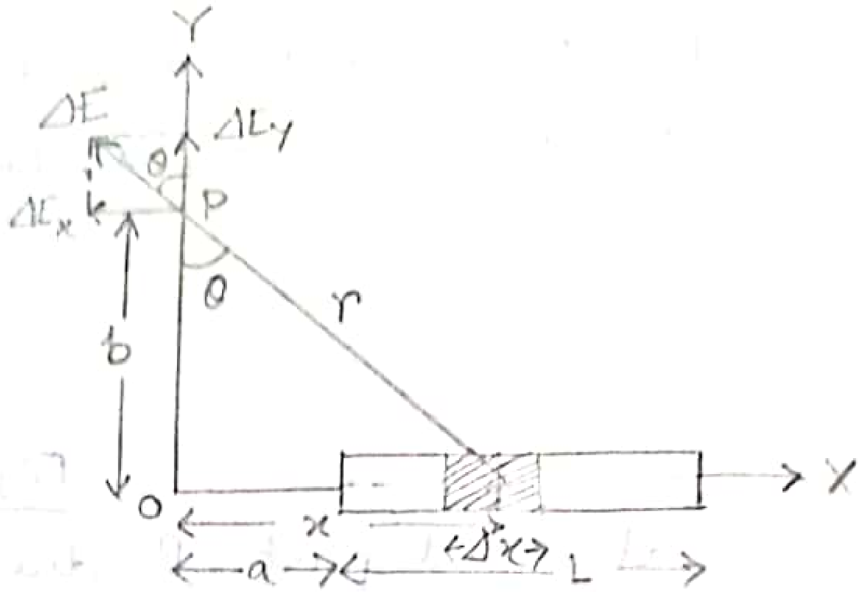
$$E = \sum \Delta E = \sum \frac{\lambda}{4\pi\epsilon_0}\frac{\Delta x}{x^2} \longrightarrow (1)$$

If Δx is sufficiently small, we can get

$$E = \int_a^{(a+L)} \frac{\lambda}{4\pi\epsilon_0}\frac{dx}{x^2} = \frac{\lambda}{4\pi\epsilon_0}\int_a^{a+L}\frac{dx}{x^2}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[-\frac{1}{x}\right]_a^{a+L}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[-\frac{1}{a+L}+\frac{1}{a}\right] = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{a}-\frac{1}{a+L}\right] \longrightarrow (2)$$



ii)

Fig. 2

Let us calculate field at point P as shown in Fig. 2.

from Fig. 2, $\quad \overrightarrow{\Delta E} = \overrightarrow{\Delta E_y} + \overrightarrow{\Delta E_x}$

Now, $\quad \Delta E_y = \Delta E \cos\theta$, along OP.

$$\therefore E_y = \sum \Delta E \cos\theta = \sum \frac{\lambda \Delta x \cos\theta}{4\pi\epsilon_0 r^2}$$

$$= \int_a^{a+L} \frac{\lambda dx}{4\pi\epsilon_0} \frac{\cos\theta}{r^2}$$

$$= \int_a^{a+L} \frac{\lambda dx}{4\pi\epsilon_0} \frac{b}{r^3} \qquad \left[\because \cos\theta = \frac{b}{r}\right]$$

$$= \int_a^{a+L} \frac{\lambda b}{4\pi\epsilon_0} \cdot \frac{dx}{(x^2+b^2)^{3/2}} \qquad \left[\because r^2 = x^2 + b^2 \Rightarrow r = (x^2+b^2)^{1/2}\right]$$

(substitute $x = b\tan\theta$)

$$= \frac{\lambda b}{4\pi\epsilon_0}\left[\frac{1}{b^2}\,\frac{x}{(x^2+b^2)^{1/2}}\right]_a^{a+L}$$

$$\Rightarrow E_y = \frac{\lambda}{4\pi\epsilon_0 b}\left[\frac{a+L}{\{(a+L)^2+b^2\}^{1/2}} - \frac{a}{(a^2+b^2)^{1/2}}\right] \rightarrow 3$$

In the similar way,

$$E_x = \sum \Delta E \sin\theta = \int_a^{a+L} \frac{\lambda}{4\pi\epsilon_0}\,\frac{x\,dx}{(x^2+b^2)^{3/2}}$$

substitute $x = b\tan\theta$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[-\frac{1}{(x^2+b^2)^{1/2}}\right]_a^{a+L}$$

$$\Rightarrow E_x = \frac{\lambda}{4\pi\epsilon_0}\left[-\frac{1}{\{(a+L)^2+b^2\}^{1/2}} + \frac{1}{(a^2+b^2)^{1/2}}\right] \rightarrow 4$$

Case I

If $a = -\frac{L}{2}$, then $E_x = 0$. That means on the points lying on the $\perp$r bisector of the rod, $E_x = 0$.

Case II

If $b >> L$, i.e., point P is very far from the rod, then we can write $[(a+L)^2 + b^2] \approx (a^2+b^2) \approx b^2$.

$$\therefore E_y = \frac{\lambda}{4\pi\epsilon_0 b} \times \frac{L}{b} = \frac{q}{4\pi\epsilon_0 b^2} \quad \text{and} \quad E_x = 0$$

This relation is same as due to point charge when viewed from far away.

Electric field due to a uniformly charged ring (or loop):

let us consider a ring of radius 'a' carrying a uniform linear charge density λ. The electric field at point 'p' on the axis of the ring at a distance

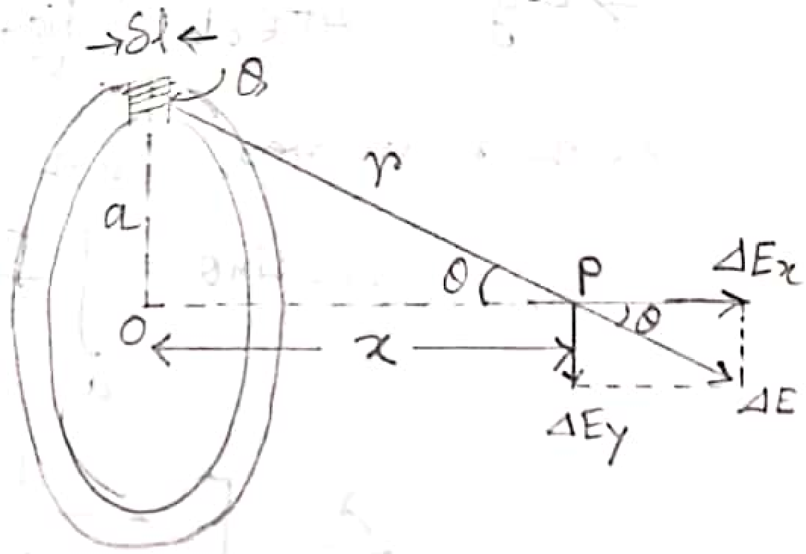


fig. 1

'x' from its centre due to the charge on the length element δl is

$$\Delta E = \frac{1}{4\pi\epsilon_0} \frac{\lambda \delta l}{r^2} \longrightarrow (1), \text{ along } QP.$$

The y-component ΔE_y cancels with the contribution due to the oppositely placed element of the ring and the net y-component thus equal to zero.

The resultant field E will be

$$E = \sum \Delta E_x = \sum \Delta E \cos\theta$$

$$= \sum \frac{1}{4\pi\epsilon_0} \frac{\lambda \delta l}{r^2} \cos\theta$$

$$= \sum \frac{1}{4\pi\epsilon_0} \frac{\lambda \delta l\, x}{r^3} \quad \left(\because \cos\theta = \frac{x}{r^3}\right)$$

$$= \sum \frac{1}{4\pi\epsilon_0} \frac{\lambda \delta l\, x}{(x^2+a^2)^{3/2}} \quad \left(\because x^2+a^2 = r^2\right)$$

$$= \int_0^l \frac{1}{4\pi\epsilon_0} \frac{x\lambda\, dl}{(x^2+a^2)^{3/2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \frac{x}{(x^2+a^2)^{3/2}} \int_0^l dl$$

$$= \frac{\lambda}{4\pi\epsilon_0} \frac{x \times 2\pi a}{(x^2+a^2)^{3/2}} \quad \left(\because \int_0^l dl = l = 2\pi a\right)$$

$$\therefore \boxed{E = \frac{1}{4\pi\epsilon_0} \frac{qx}{(x^2+a^2)^{3/2}}} \quad \left(\because 2\pi a \times \lambda = q\right)$$

If $x >> a$, then $(x^2+a^2) \approx x^2$ and the above equation will reduce to

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{x^2}$$

The above eqn is exactly same as due to a point charge.

Electric field due to a uniformly charged disc :

Let us calculate the field due to a charged disc of surface charge density σ at a distance x from its centre lying on its axis.

The field at point 'p' due to the charge

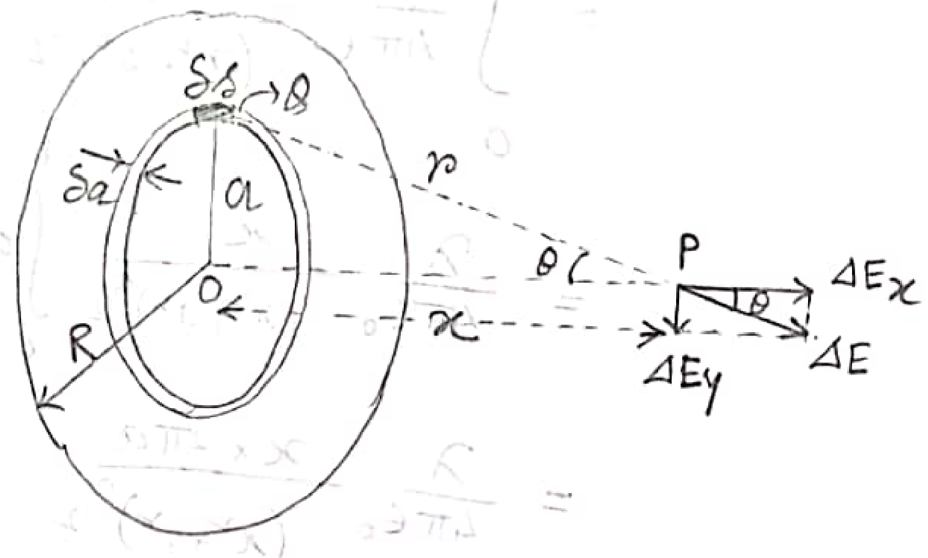


fig.1

element 'δs' on the annular ring of radius 'a' and thickness 'δa' is given by

$$\Delta E = \frac{1}{4\pi\epsilon_0}\frac{\sigma \delta s}{r^2}, \text{ along } QP.$$

The y-component ΔE_y cancels with the contribution due to the oppositely ~~charged~~ placed element of the disc and the net y-component thus equal to zero. Hence the resultant field dE due to annular ring of radius a will be

$$dE = \sum \Delta E_x = \sum \Delta E \cos\theta$$

$$= \sum \frac{1}{4\pi\epsilon_0}\frac{\sigma \delta s}{r^2}\cos\theta$$

$$= \int_0^S \frac{1}{4\pi\epsilon_0} \frac{\sigma\, ds}{r^2} \cos\theta = \frac{\sigma}{4\pi\epsilon_0} \frac{s}{r^2} \cos\theta$$

$$= \frac{\sigma}{4\pi\epsilon_0} \int_0^R \frac{2\pi a\, da}{r^2} \cos\theta \quad \left(\because \cancel{ds}\ s = 2\pi a\, da\right)$$

$$= \frac{\sigma}{2\epsilon_0} \frac{a\, da}{x^2+a^2} \times \frac{x}{(x^2+a^2)^{1/2}} \quad \left(\because x^2+a^2 = r^2 \ \& \cos\theta = \frac{x}{r}\right)$$

$$\Rightarrow dE = \frac{\sigma}{2\epsilon_0} \frac{ax\, da}{(x^2+a^2)^{3/2}}$$

$\therefore$ Net field due to the entire disc is

$$E = \int_0^E dE = \frac{\sigma x}{2\epsilon_0} \int_0^R \frac{a\, da}{(x^2+a^2)^{3/2}}$$ Substitute $a = x\tan\theta$

$$= \frac{\sigma x}{2\epsilon_0} \left[-\frac{1}{(x^2+a^2)^{1/2}} \right]_0^R$$

$$= \frac{\sigma x}{2\epsilon_0} \left[-\frac{1}{(x^2+R^2)^{1/2}} + \frac{1}{x} \right]$$

$$\therefore \boxed{E = \frac{\sigma}{2\epsilon_0} \left[1 - \frac{x}{(x^2+R^2)^{1/2}} \right]}$$

Electric potential due to a uniformly charged wire:

Let us consider a uniformly charged wire of linear charge density λ. Electric field at point 'p' will be

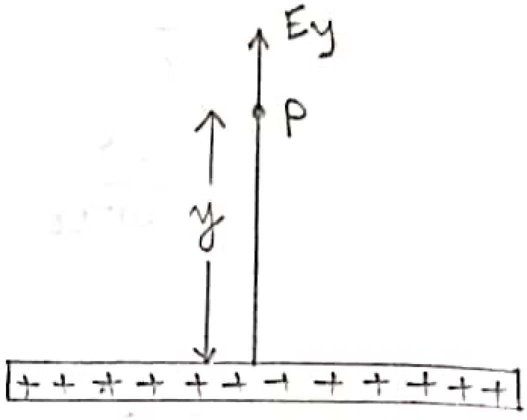


fig-1

$$E = \frac{\lambda}{2\pi\epsilon_0 y} \longrightarrow (1)$$

∵ Electric field is given by −ve gradient of potential, we have,

$$E = -\frac{dV_y}{dy}$$

$$\Rightarrow dV_y = -\vec{E}\cdot\vec{dy} = -E\,dy\cos 0^\circ = -E\,dy$$

$$\Rightarrow V_y = -\int_{\alpha}^{y} \vec{E}\cdot\vec{dy} = -\int_{\alpha}^{y} E\,dy$$

$$= -\int_{\alpha}^{y} \frac{\lambda}{2\pi\epsilon_0 y}\,dy = -\frac{\lambda}{2\pi\epsilon_0}\left[\log_e y\right]_{\alpha}^{y}$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\left(\log_e y - \log_e \alpha\right)$$

$$= \frac{\lambda}{2\pi\epsilon_0}\left(\log_e \alpha - \log_e y\right) = \frac{\lambda}{2\pi\epsilon_0}\log_e\left(\frac{\alpha}{y}\right)$$

$$\Rightarrow \boxed{V_y = \alpha}$$

Thus the electric potential due to a uniformly charged wire of infinite length is infinity everywhere. But in practical problems we are interested in potential difference betn positions with finite separation and never in absolute potentials. In this case our choice of infinity for the point of zero potential was arbitrary. We can, however, set $V=0$ at some arbitrary y_0 as

$$V_y = \frac{\lambda}{2\pi\epsilon_0}\log_e\left(\frac{y_0}{y}\right) = \frac{\lambda}{2\pi\epsilon_0}\log_e y_0 - \frac{\lambda}{2\pi\epsilon_0}\log_e y$$

$$\Rightarrow V_y = -\frac{\lambda}{2\pi\epsilon_0}\log_e y + C \longrightarrow (2)$$

where, $C = \frac{\lambda}{2\pi\epsilon_0}\log_e y_0 \rightarrow$ a constant ($\because y_0$ – const.)

Electric potential due to a uniformly charged ring:

Let us find the electric potential at a point 'P' on the axis of a uniformly charged circular ring of linear charge density λ.

Now potential at 'P' due to length element 'δl' is

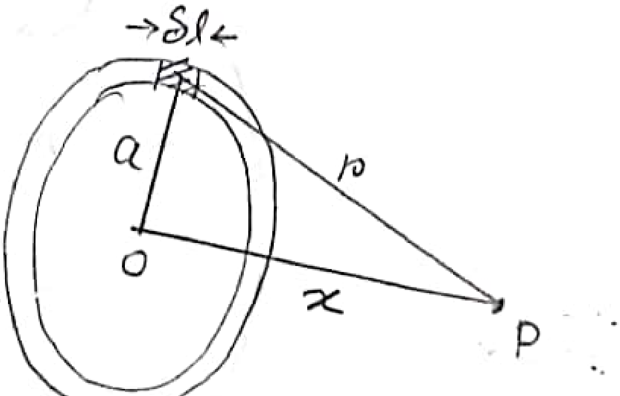


Fig. 1

$$\Delta V = \frac{1}{4\pi\epsilon_0}\frac{\Delta q}{r}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\lambda\,\delta l}{(x^2+a^2)^{1/2}}$$

Thus the potential due to the whole ring is

$$V=\int_0^V dV = \int_0^l \frac{\lambda}{4\pi\epsilon_0}\frac{dl}{(a^2+x^2)^{1/2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\frac{l}{(x^2+a^2)^{1/2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\frac{2\pi a}{(x^2+a^2)^{1/2}}$$

$$\therefore \boxed{V = \frac{\lambda}{2\epsilon_0}\frac{a}{(x^2+a^2)^{1/2}}}$$

Potential due to a uniformly charged disc :

Let us find the electric potential at any point 'p' on

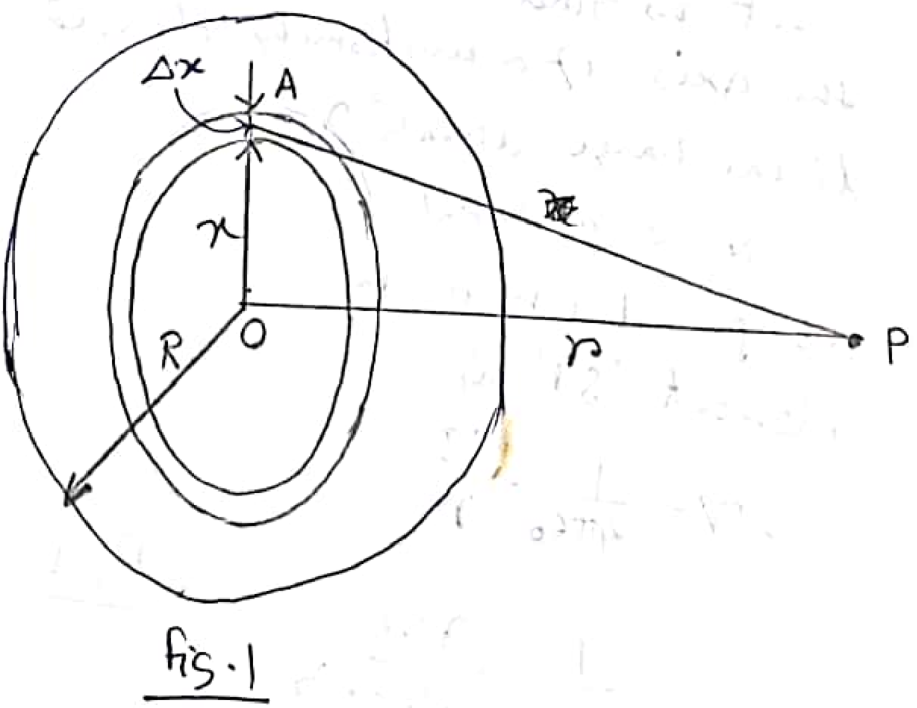


fig. 1

the axis of a uniformly charged circular disc, having surface charge density σ, at a distance r from the centre of the disc.

Now potential at 'p' due to annular ring of width Δx is

$$\Delta V = \frac{1}{4\pi\epsilon_0}\frac{\Delta q}{AP} = \frac{1}{4\pi\epsilon_0}\frac{2\pi x\Delta x\sigma}{(r^2+x^2)^{1/2}}$$

$(\because \Delta q = ds \times \sigma = 2\pi x \Delta x \sigma$
$ds \rightarrow$ area of ring $)$

Thus the potential due to the charge on the whole disc is

$$V = \int_0^V dV = \frac{\sigma}{2\epsilon_0}\int_0^R \frac{x\,dx}{(x^2+r^2)^{1/2}}.$$

$\sigma = \frac{q}{\text{area}}$
$\Rightarrow q = \sigma \times \text{area}$
$= \sigma \times \pi R^2$

substitute
$x^2+r^2 = \cancel{\sin^2\theta}\ p^2$

$$= \frac{\sigma}{2\epsilon_0}\left[(x^2+r^2)^{1/2}\right]_0^R$$

$$\Rightarrow \boxed{V = \frac{\sigma}{2\epsilon_0}\left[(R^2+r^2)^{1/2} - r\right]} \rightarrow (1)$$

At the centre 'O'

$$V = \frac{\sigma R}{2\epsilon_0} \qquad (\text{as } r = 0)$$

using Binomial theorem for $r \gg R$, eqn (1) can be written as

$$V = \frac{\sigma}{2\epsilon_0}\left[r + \frac{R^2}{2r} - r\right] = \frac{\sigma\pi R^2}{4\pi\epsilon_0 r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

$$\left[(r^2+R^2)^{1/2} = r\left(1+\frac{R^2}{r^2}\right)^{1/2} \approx r\left(1+\frac{1}{2}\frac{R^2}{r^2}\right) = r + \frac{R^2}{2r}\right]$$

Divergence of electric field :

The divergence of a vector is concerned with the net outward flow of some physical quantity, e.g., electric flux through the surface area of unit volume element in a vector field.

The divergence of electric field at a point in space is equal to the charge density divided by the permittivity of space.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

In a charge free region of space where $\rho = 0$, we get

$$\vec{\nabla} \cdot \vec{E} = 0$$

Curl of electric field :

The curl or rotation of a vector is concerned with the rotation of a vector, or with rotational fields. It arises in problems where a line integral of a vector round a closed path is related to the flux through the surface enclosed by the path of integration.

The electric field eqn is given by

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \quad \rightarrow (1)$$

, $\vec{A} \rightarrow$ magnetic vector potential

We know, $\vec{B} = \vec{\nabla} \times \vec{A} \rightarrow (2)$ $\quad \vec{B} \rightarrow$ magnetic flux density

Taking curl of eqn (1),

$$\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times\vec{\nabla}\phi - \partial_t(\nabla\times\vec{A})$$

$$\Rightarrow \vec{\nabla}\times\vec{E} = 0 - \frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$ which is the Faraday's law of induction.

In electrostatic,

$$\vec{E} = -\vec{\nabla}\phi$$

$$\therefore \vec{\nabla}\times\vec{E} = \vec{\nabla}\times(-\vec{\nabla}\phi) = -\vec{\nabla}\times\vec{\nabla}\phi = 0$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = 0}$$ ($\because$ divergence of gradient of a scalar field is zero)

(1)

ELECTROSTATIC ENERGY

Electrostatic energy: If we consider two charges q_1 and q_2 a distance 'r' apart, then there is some energy in the system. This energy is due to the fact that certain amount of work is necessary to bring the charges together if the charges are of same sign. This energy is known as the electric potential energy in the system of (q_1+q_2) charges. This potential energy is termed as electrostatic energy.

Electrostatic energy of an assembly of point charges:

Let us consider N no. of point charges infinitely separated from one another. We first place the charge q_1 in its position $\vec{r_1}$. This will require no work because q_1 is not interacting with any other charges. Next we bring a charge q_2 at its location $\vec{r_2}$. This requires an amount of work and it is given by

$$V_1(2)\, q_2 = \frac{q_1 q_2}{4\pi\epsilon_0 r_{21}}$$

where $r_{21} = |\vec{r_2} - \vec{r_1}| \rightarrow$ distance of q_2 from q_1.

$V_1(2) \rightarrow$ the potential at the position of q_2 due to charge q_1

$$\text{i.e., } V_1(2) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_{21}}$$

Then we place the charge q_3 at its location $\vec{r_3}$. Work is now done against the fields of q_1 and q_2. The work done is

$$[V_1(3) + V_2(3)]\, q_3 = \frac{q_3}{4\pi\epsilon_0}\left[\frac{q_1}{r_{31}} + \frac{q_2}{r_{32}}\right]$$

Continuing this process of bringing one charge after another we find that the work done in bringing the jth charge q_j is

$$[V_1(j) + V_2(j) + \cdots + V_{j-1}(j)]\, q_j$$

$$= \frac{q_j}{4\pi\epsilon_0}\left[\frac{q_1}{r_{j1}} + \frac{q_2}{r_{j2}} + \cdots + \frac{q_{j-1}}{r_{j(j-1)}}\right]$$

The electrostatic energy U of the assembled N

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charges is obtained by summing up the individual works done in bringing them one at a time. Thus

$$U = \sum_{j=1}^{N}\left(\frac{q_j}{4\pi\epsilon_0}\sum_{i=1}^{j-1}\frac{q_i}{r_{ji}}\right) = \frac{1}{2}\sum_{j=1}^{N}\sum_{i=1}^{N'}\frac{q_i q_j}{4\pi\epsilon_0 r_{ji}} \longrightarrow (1)$$

The ~~tast~~ factor $\frac{1}{2}$ appears in the last step to avoid double counting the interaction betn each pair of charges. The prime on the second summation means that the term with $i=j$ is excluded.

The potential at the location of the j^{th} point charge due to all other charges is

$$V(j) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N'}\frac{q_i}{r_{ji}} \longrightarrow (2)$$

Hence the electrostatic energy due to a system of point charges is

$$\boxed{U = \frac{1}{2}\sum_{j=1}^{N} q_j V(j)} \longrightarrow (3)$$

<u>Electrostatic energy of a uniformly charge sphere:</u>

Let us consider a uniformly charged sphere of volume charge density ρ. The electrostatic energy for this system is

$$U = \frac{1}{2}\int_{\tau}\rho(\vec{r})\,V(\vec{r})\,d\tau \longrightarrow (1) \quad (\tau \rightarrow \text{volume})$$

The differential from of Gauss's law gives

$$\vec{\nabla}\cdot\vec{D} = \rho$$

$$\therefore (1) \Rightarrow \quad U = \frac{1}{2}\int_{\tau} V(\vec{\nabla}\cdot\vec{D})\,d\tau \longrightarrow (2)$$

let us use the vector identity

$$\vec{\nabla}\cdot(V\vec{D}) = \vec{\nabla}V\cdot\vec{D} + V(\vec{\nabla}\cdot\vec{D})$$

$$\therefore (2) \Rightarrow \quad U = \frac{1}{2}\int_{\tau}\vec{\nabla}\cdot(V\vec{D})\,d\tau - \frac{1}{2}\int_{\tau}\vec{\nabla}V\cdot\vec{D}\,d\tau \rightarrow (3)$$

$$\Rightarrow U = \frac{1}{2}\oint_{S}(V\vec{D})\cdot\hat{n}\,ds + \frac{1}{2}\int_{\tau}\vec{E}\cdot\vec{D}\,d\tau \rightarrow (4)$$

$$\left(\because \oint_{S}\hat{D}\cdot\hat{n}\,ds = \int_{\tau}\vec{\nabla}\cdot\vec{D}\,d\tau \;\; \& \;\; \vec{E} = -\vec{\nabla}V\right)$$

where s is the closed surface enclosing volume V. The integral over S falls off as $\frac{1}{r}$ and vanishes when S recedes to infinity.

$$\therefore (4) \Rightarrow \boxed{U = \frac{1}{2}\int_{\tau}\vec{D}\cdot\vec{E}\,d\tau} \longrightarrow (5)$$

Eqn (5) may be visualized as the electrostatic energy being stored in the electric field with an energy of $\frac{1}{2}\vec{D}\cdot\vec{E}$ per unit volume. On this basis, we find that the energy density in an electrostatic field is

$$U = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}\frac{D^2}{\epsilon} = \frac{1}{2}\epsilon E^2 \rightarrow (6)$$

$$(\because \vec{D} = \epsilon\vec{E})$$

$$\Rightarrow \boxed{U = \frac{1}{2}\epsilon E^2}$$

Laplace's and Poisson's equation:

These are the fundamental differential equations satisfied by the potential V. We have,

$$\vec{E} = -\vec{\nabla} V \longrightarrow (1)$$

From Gauss's law

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \longrightarrow (2)$$

$$\Rightarrow \vec{\nabla} \cdot (-\vec{\nabla} V) = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow -\nabla^2 V = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}} \longrightarrow (3)$$

where ∇^2 is called the Laplacian operator and eqn (3) is called the Poisson's eqn.

If $\rho = 0$ in some region of space, then in that region Poisson's eqn reduces to

$$\boxed{\nabla^2 V = 0} \longrightarrow (4)$$

Eqn (4) is known as Laplace's eqn.

Different forms of ∇^2

i) Rectangular cartesian co-ordinates:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

ii) Spherical polar coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

iii) Cylindrical Co-ordinates:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

<u>Boundary conditions at the interface of two conducting media:</u>

For steady current, we have

$$\vec{\nabla}\cdot\vec{J} = 0 \Rightarrow \int_V \vec{\nabla}\cdot\vec{J}\, dv = 0 \Rightarrow \oint \vec{J}\cdot d\vec{s} = 0$$

$$\Rightarrow \oint_S \vec{J}\cdot\hat{n}\, ds = 0 \longrightarrow (1) \quad (\text{where } d\vec{s} = ds\,\hat{n})$$

This is analogous to the relationship for the displacement vector $\vec{D}$.

We have,

$$J_{1n} = J_{2n} \longrightarrow (2)$$

where $J_{1n} \rightarrow$ Normal component of the current density in Medium 1

$J_{2n} \rightarrow$ - - - - medium 2.

Eqn (2) shows that the normal components of the electric field at the interface, E_{1n} and E_{2n} are related by

$$\sigma_{c1} E_{1n} = \sigma_{c2} E_{2n} \longrightarrow (3)$$

where σ_{c1} & σ_{c2} are the conductivities of media 1 and 2 respectively.

If J_{1t} and J_{2t} are the tangential components of the current density at the interface in media 1 and 2 respectively, then

$$\frac{J_{1t}}{\sigma_{c1}} = \frac{J_{2t}}{\sigma_{c2}} \longrightarrow (4)$$

Boundary conditions for dielectrics :

Boundary conditions tell us how the electric field $\vec{E}$ and the displacement $\vec{D}$ behave when we move from one medium to another.

fig.1 : Boundary condition on the normal comp. of $\vec{D}$

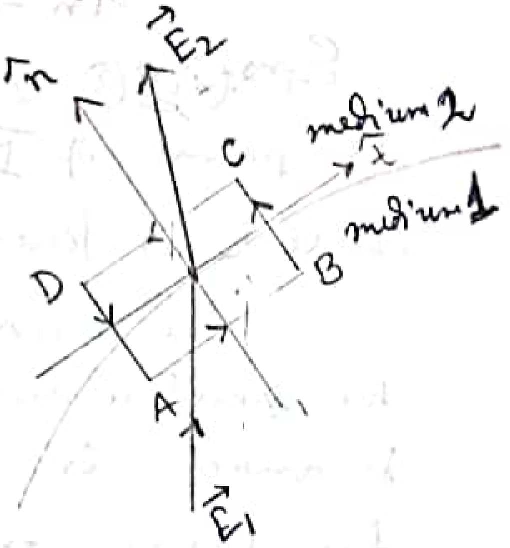


fig.2 : Boundary condition on the tangential comp. of $\vec{E}$

Let us consider the interface between two media 1 and 2, and imagine a Gaussian cylinder as shown in Fig.1. Let ΔS be the surface area cut by the ~~the~~ Gaussian cylinder and σ be the surface charge density of free charge on the interface.

Applying Gauss's law, we have,

$$\vec{D_2}\cdot\hat{n}\Delta S - \vec{D_1}\cdot\hat{n}\Delta S = \sigma\Delta S \longrightarrow ①$$

where $\hat{n}$ is the ~~pointing~~ unit vector pointing from medium 1 to medium 2. Flux through the curved surface is neglected due to negligible height of the cylinder.

$$① \Rightarrow \quad \vec{D_2}\cdot\hat{n} - \vec{D_1}\cdot\hat{n} = \sigma \longrightarrow ②$$

$$\Rightarrow \boxed{D_{2n} - D_{1n} = \sigma} \longrightarrow ③$$

where $D_{2n} \rightarrow$ normal component of D_2

$D_{1n} \rightarrow$ normal component of D_1

Eqns ② OR ③ gives the boundary condition on the normal component of $\vec{D}$. The boundary condition may be stated as follows:

The discontinuity in the normal component of the dielectric displacement in moving from one medium to another is predicted by the surface density of free charge on the interface between the media. If there is no free charge on the interface, the normal component of $\vec{D}$ is continuous across it.

The line integral of electric field $\vec{E}$ around any closed path is zero. From Fig.2, neglecting lengths of sides BC and DA, we have,

$$\vec{E_1}\cdot\vec{\Delta l} - \vec{E_2}\cdot\vec{\Delta l} = 0 \quad (\because AB = CD = \Delta l)$$

$$\Rightarrow \vec{E_1}\cdot\hat{t}\,\Delta l - \vec{E_2}\cdot\hat{t}\,\Delta l = 0 \quad (\hat{t} \rightarrow \text{unit vector along the tangent to the interface})$$

$$\Rightarrow \vec{E_1}\cdot\hat{t} = \vec{E_2}\cdot\hat{t} \rightarrow (4)$$

$$\Rightarrow \boxed{E_{1t} = E_{2t}} \rightarrow (5)$$

, $E_{1t} \rightarrow$ tangential component of electric field E_1

$E_{2t} \rightarrow$ . . . . . E_2

Eqn (4) or (5) gives the boundary condition on the tangential component of $\vec{E}$. It may be stated as follows:

The tangential component of the electric field is continuous across the interface betn two media.

The results stated above hold for any two media. If one of the media, say, medium 1 is a conductor, then $\vec{E_1}$ and $\vec{D_1}$ are zero. Then

(3) $\Rightarrow D_{2n} = \sigma$ & (5) $\Rightarrow E_{2t} = 0$

Thus the normal component of the dielectric displacement in the dielectric is equal to the surface density of free charge on the interface when the dielectric is in contact with a conductor, Also the electric field in the dielectric is normal to the interface.

Uniqueness theorem :

Two solutions of Laplace's eqn obeying the same boundary conditions differ at best by a constant.

Q. State and prove uniqueness theorem.

Statement: Two solutions of Laplace's equation obeying the same boundary conditions differ at best by a constant.

Proof: Let us assume that ϕ_1 and ϕ_2 are the two solns of Laplace's eqn in a volume V exterior to the surfaces $S_1, S_2, \cdots, S_n$ of the different conductors and bounded on the outside by a surface S. Let ϕ_1 & ϕ_2 satisfy the same boundary conditions on the various surfaces, i.e., $S, S_1, S_2, \cdots, S_n$.

Let $\phi = \phi_1 - \phi_2$

then $\nabla^2\phi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0$

(since both ϕ_1 and ϕ_2 satisfy Laplace's equation)

If Dirichlet condition is satisfied, we have $\phi_1 = \phi_2$ on the bounding surfaces.

Hence $\phi = 0$

If Neumann condition is satisfied, $\frac{\partial \phi_1}{\partial n} = \frac{\partial \phi_2}{\partial n}$ on the bounding surfaces.

Hence $\frac{\partial \phi}{\partial n} = 0$ or $\vec{\nabla}\phi \cdot \hat{n} = 0$ on the boundaries. $\hat{n}$ being the unit vector along the outward normal to the surface in question.

Applying Gauss's divergence theorem to the vector $\phi\vec{\nabla}\phi$, we obtain,

$$\int_V \vec{\nabla}\cdot(\phi\vec{\nabla}\phi)\,dv = \int_{S+S_1+S_2+\cdots+S_n} \phi\vec{\nabla}\phi\cdot\hat{n}\,ds \quad \rightarrow (1)$$

$$= 0$$

($\because$ $\phi = 0$ on the boundaries for ~~Dirichlet~~ Dirichlet condition and $\vec{\nabla}\phi . \hat{n} = 0$ on the boundaries for Neumann condition.)

we have,

$$\vec{\nabla}.(\phi \vec{\nabla}\phi) = \phi(\vec{\nabla}.\vec{\nabla}\phi) + (\vec{\nabla}\phi)(\vec{\nabla}\phi)$$

$$= \phi \nabla^2\phi + |\vec{\nabla}\phi|^2$$

$$\Rightarrow \vec{\nabla}.(\phi\vec{\nabla}\phi) = |\vec{\nabla}\phi|^2 \quad (\because \nabla^2\phi = 0 \text{ in the region } V) \rightarrow (2)$$

using (2) in (1), we have,

$$\int_V |\vec{\nabla}\phi|^2 dV = 0 \longrightarrow (3)$$

Now $|\vec{\nabla}\phi|^2$, being a perfect square, is either +ve or zero. But since its integral vanishes, $\vec{\nabla}\phi$ must be zero at each point in V.

$$\therefore \vec{\nabla}\phi = 0$$

$$\Rightarrow \phi = c, \quad c \rightarrow \text{const.}$$

$$\Rightarrow \boxed{\phi_1 - \phi_2 = c} \longrightarrow (4)$$

Eqn (4) proves the uniqueness theorem.

Solutions of Laplace's equation in one dimension:

In the case where ϕ is a fn of the single rectangular coordinate x, Laplace's eqn becomes

$$\frac{d^2\phi}{dx^2} = 0 \longrightarrow (1)$$

Soln of (1) is $\boxed{\phi = ax + b} \longrightarrow (2)$

$\frac{d}{dx}\left(\frac{d\phi}{dx}\right) = 0$

$\Rightarrow \frac{d\phi}{dx} = a$

$\Rightarrow d\phi = a\,dx$

$\Rightarrow \int d\phi = \int a\,dx + b$

$\Rightarrow \phi = ax + b$

where a and b are the constants to be determined from the boundary conditions.

For a spherical ~~symmetry~~ geometry where ϕ is a fn of 'r', the Laplace's eqn is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 0 \longrightarrow (3)$$

$$\Rightarrow \frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 0 \quad (\because r \neq 0)$$

Integrating

$$\Rightarrow r^2\frac{d\phi}{dr} = a, \quad a \to \text{const. of integration}$$

$$\Rightarrow \frac{d\phi}{dr} = a/r^2$$

$$\Rightarrow d\phi = \frac{a}{r^2}\,dr$$

$$\Rightarrow \int d\phi = \int \frac{a}{r^2}\,dr + b, \quad b \to \text{const. of integration}$$

$$\Rightarrow \boxed{\phi = -\frac{a}{r} + b} \longrightarrow (4)$$

The constants a and b are to be chosen to fit the boundary conditions.

In cylindrical co-ordinates, when ϕ is a fn of 'r' only, Laplace's eqn becomes –

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0 \longrightarrow (5)$$

$$\Rightarrow \frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0 \quad (\because r \neq 0)$$

$$\Rightarrow r\frac{d\phi}{dr} = a \quad \text{(integrating)}, \ a \to \text{const. of integration}$$

$$\Rightarrow \frac{d\phi}{dr} = \frac{a}{r}$$

$$\Rightarrow \int d\phi = \int \frac{a}{r}\, dr + b, \quad b \to \text{constant of integration}$$

$$\Rightarrow \phi = a \ln r + b \longrightarrow (6)$$

Let, at $r = r_0$, $\phi = 0$.

$$\therefore (6) \Rightarrow 0 = a \ln r_0 + b \Rightarrow b = -a \ln r_0$$

$$\therefore (6) \Rightarrow \phi = a \ln r - a \ln r_0 = a \ln\left(\frac{r}{r_0}\right) \longrightarrow (7)$$

$r_0 \to$ another constant.

These constants a, b or r_0 can be determined from the boundary conditions.

Electric ~~field~~ potential and intensity inside an infinite parallel plate capacitor :-

Let ϕ_1 and ϕ_2 be the potentials of two parallel plates of infinite dimension. Let the plates be located at $x = 0$ and $x = d$ as shown in Fig. 1. In the space betn the plates there is no charge. Hence Laplace's eqn holds and is given by

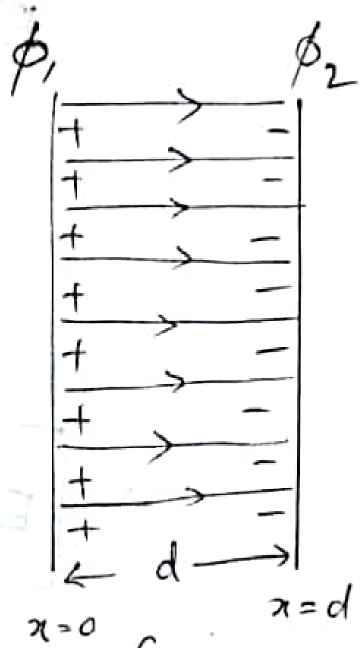


Fig. 1

$$\frac{d^2\phi}{dx^2} = 0 \longrightarrow ①$$

Soln of ① is

$$\phi = ax + b \longrightarrow ②$$

where a, b are the constants to be determined from the boundary conditions.

Now at $x = 0$, $\phi = \phi_1$ and at $x = d$, $\phi = \phi_2$

$\therefore$ ② $\Rightarrow$ $\boxed{\phi_1 = b}$ and $\phi_2 = ad + b = ad + \phi_1$

$$\Rightarrow \boxed{a = \frac{\phi_2 - \phi_1}{d}}$$

$\therefore$ ② $\Rightarrow$ $\boxed{\phi = \frac{\phi_2 - \phi_1}{d} x + \phi_1}$ The potential inside the plates of the capacitor.

The electric field will be

$$E = -\frac{d\phi}{dx} = -\frac{d}{dx}\left[\frac{\phi_2 - \phi_1}{d} x + \phi_1\right]$$

$$= -\frac{\phi_2 - \phi_1}{d} - \frac{d\phi_1}{dx}$$

$$= \frac{\phi_1 - \phi_2}{d} - 0 \quad (\because \phi_1 = b \to \text{a const.})$$

$\Rightarrow$ $\boxed{E = \frac{\phi_1 - \phi_2}{d}}$ The electric field inside the parallel plate capacitor.

Electric potential and intensity inside spherical capacitor:

Let a charge '+q' be placed on the inner shell. The outer shell will be induced by '-q'.

To determine the electric field E at a distance 'r' from 'O' ($a_1 < r < a_2$), we consider a spherical Gaussian surface of radius r.

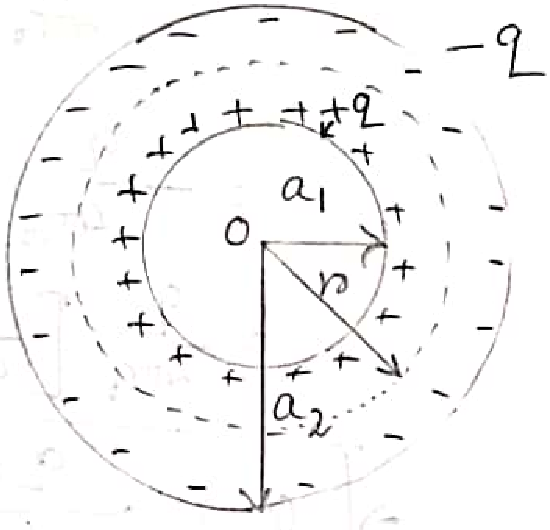


Fig. 1 : A spherical capacitor

Now according to Gauss's law,

$$\oint_S \vec{E}\cdot\vec{ds} = \frac{q}{\epsilon_0}$$

$$\Rightarrow \int_S E\,dS = \frac{q}{\epsilon_0} \quad (\because \text{ angle bet}^n\ \vec{E}\ \&\ \vec{ds} \text{ is } 0^\circ)$$

$$\Rightarrow ES = \frac{q}{\epsilon_0}$$

$$\Rightarrow E \times 4\pi r^2 = \frac{q}{\epsilon_0}$$

$$\Rightarrow \boxed{E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}}$$ The electric intensity the spherical capacitor

Let ϕ be the potential difference bet^n the spherical shells.

$$\therefore E = -\frac{d\phi}{dr}$$

$$\Rightarrow d\phi = -E\,dr = -\frac{q}{4\pi\epsilon_0 r^2}dr$$

$$\therefore \phi = \int_0^\phi d\phi = -\frac{q}{4\pi\epsilon_0}\int_{a_2}^{a_1}\frac{1}{r^2}dr$$

$$= -\frac{q}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_{a_2}^{a_1}$$

$$= -\frac{q}{4\pi\epsilon_0}\left[-\frac{1}{a_1}+\frac{1}{a_2}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{a_1}-\frac{1}{a_2}\right]$$

$$\Rightarrow \boxed{\phi = \frac{q(a_2-a_1)}{4\pi\epsilon_0 a_1 a_2}}$$ The potential inside the spherical capacitor.

Electric potential and intensity due to long and ~~[illegible]~~ uniformly charged conducting wire:

(goto II step first) Let us consider a uniformly charged conductor of linear charge density λ. Let us calculate the potential difference betn the points A and B at a distance r_A and r_B from the wire respectively.

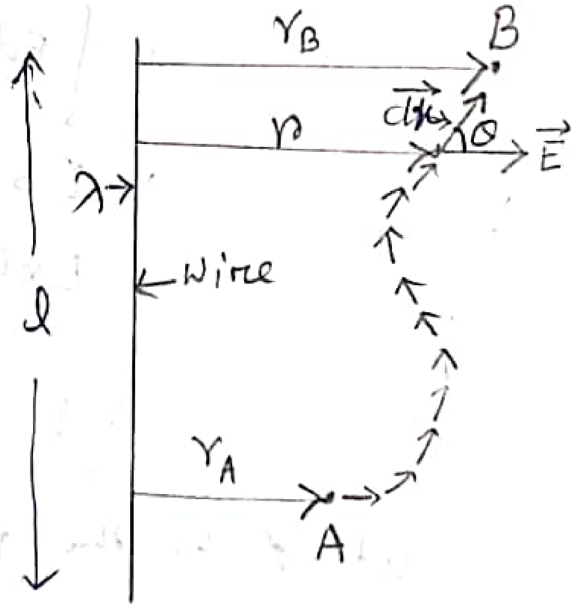


Fig. 1

$$V_B - V_A = -\int_A^B \vec{E}\cdot\vec{dr}$$

$$= -\int_A^B E\,dr\cos\theta$$

$$\vec{E} = -\frac{dv}{dr}\hat{r}$$

$$\Rightarrow dv = -\int \vec{E}\cdot dr$$

$$V_B - V_A = -\int_{r_A}^{r_B} \frac{\lambda}{2\pi\epsilon_0 x}\,dx$$

$$= -\int_A^B \frac{\lambda}{2\pi\epsilon_0 r}\, dx\cos\theta \qquad \left(\because E = \frac{\lambda}{2\pi\epsilon_0 r}\right)$$

$$= -\int_A^B \frac{\lambda}{2\pi\epsilon_0 r}\, dr \qquad (\because dx\cos\theta = dr)$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\int_{r_A}^{r_B} \frac{dr}{r}$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\left[\ln r\right]_{r_A}^{r_B} \cancel{\epsilon_0}$$

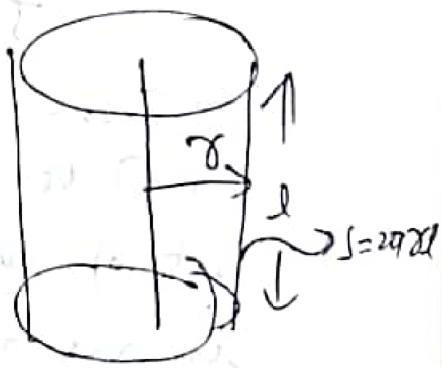
$$= -\frac{\lambda}{2\pi\epsilon_0}\left[\ln r_B - \ln r_A\right]$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_B}{r_A}$$

$$\therefore \boxed{V_B - V_A = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_A}{r_B}}$$

2nd step

Gauss's law gives,



$$\oint_S \vec{E}\cdot\vec{ds} = \frac{\lambda l}{\epsilon_0}$$

$$\Rightarrow E\,S = \frac{\lambda l}{\epsilon_0}$$

$$\Rightarrow E \times 2\pi r l = \frac{\lambda l}{\epsilon_0}$$

$$\Rightarrow \boxed{E = \frac{\lambda}{2\pi\epsilon_0 r}}$$