Origin of the concept of Tensor :

The origin of the concept of tensors is indebted from the idea of "deformation of space" due to Gauss. The generalization of Differential Geometry of Gauss and Riemann leads to the "absolute differential calculus" or "Tensor calculus". A continuous transformation of a surface (like a plane sheet of paper is folded to a cylinder — a developable surface) preserving some properties, is called a deformation. In deformation of a surface, Gauss has established that the total curvature K (= product of minimum and maximum curvatures of normal sections through a normal at a point to the surface) is invariant which depends only on the coefficients E, F, G of the 1st fundamental form

$$ds^2 = E du^2 + 2F du\, dv + G dv^2$$

It is transformable to $ds^2 = dx^2 + dy^2$ that holds in Euclidean space of two dimensions if $(EG-F^2)\times K = 0$. But the fundamental property to be preserved in the transformation Geometry of the space is the metric ds^2. Finally, it can be said that the concept of transformations equipped with the property of invariance gives rise to the 'notion' — The TENSOR.
conception

Def^n of n-dimensional space :

Consider an ordered set of n-real variables, called co-ordinates $(x^1, x^2, x^3, \cdots, x^n)$. The space generated by all points corresponding to different values of the coordinates, is called the n-dimensional space and is denoted by V_n.

Riemannian metric and Riemannian space :

A differentiable manifold is defined as a continuum of points x^i $(i=1,2,\cdots,n)$ of n independent parameters called the co-ordinates of V_n. The square of the infinitesimal distance 'ds' between two adjacent points x^i and (x^i+dx^i) of V_n is defined by Riemann as

$$ds^2 = \sum_{i,j=1}^{n} g_{i,j}(x^i)\,dx^i dx^j$$

where $g = |g_{ij}| = \begin{vmatrix} g_{11} & g_{12} & \cdots & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & \cdots & g_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{n1} & g_{n2} & \cdots & \cdots & g_{nm} \end{vmatrix}$ is non-singular

This is called Riemann metric of n-dimensional and the geometry (or space) generated by the metric is called the Riemannian Geometry or space along with the introduction of the idea of distance, direction etc.

Summation Convention: The summation convention means drop of summation sign (Σ) in an expression

$$\sum_{i=1}^{n} a_i x^i = a_1 x^1 + a_2 x^2 + \cdots + a_n x^n = a_i x^i$$

which implies the sum of n-terms ranging the value of i from 1 to n in general.

Dummy Suffix:

A suffix which occurs twice in a term once in upper position (usually) and once in a lower position is called a dummy suffix. In $a_i x^i$, 'i' is a dummy suffix and it can be replaced by any other suffix with the same range of values.

Real Suffix: A suffix which occurs only once in a term is called a real suffix and it can not be changed by any other suffix like a dummy suffix. In the set of relations

$y^i = a^i_k x^k$ $(i, k = 1, 2, \cdots, n)$, 'i' is a real suffix and 'k' is a dummy suffix

Kronecker Delta :

If x^i's are n-independent functions Then

$$\frac{\partial x^i}{\partial x^j} = 0 \text{ when } i \neq j$$
$$= 1 \text{ when } i = j$$

$$\delta^i_j = \frac{\partial x^i}{\partial x^j} = \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\alpha}{\partial x^j}$$

This is denoted by δ^i_j symbolically so that

$$\delta^i_j = 1 \text{ when } i = j$$
$$= 0 \text{ when } i \neq j$$

$$\delta^\alpha_\beta = \frac{\partial \bar{x}^\alpha}{\partial \bar{x}^\beta} = \frac{\partial \bar{x}^\alpha}{\partial x^K} \frac{\partial x^K}{\partial \bar{x}^\beta}$$

Here δ^i_j is called The Kronecker Delta.

$$\bar{\delta}^i_j = \delta^i_j$$

Tensor analysis :

Tensor analysis is that part of study which is rather suitable for the mathematical formulation of natural laws in the forms ~~which~~ that are invariant with respect to different frames of reference. The tensor formulation was originated by G. Ricci (1901) and it became popular when Albert Einstein used it as a natural tool for the description of his general Theory of relativity. Tensor analysis is the

[Gregorio (1853-1925) Italy]

generaligation of vector analysis as can be shown by considering a vector function $f(\vec{r})$ of a vector $\vec{r}$. This vector function is continuous for $\vec{r} = \vec{r}_0$ if

$$\lim_{\vec{r} \to \vec{r}_0} f(\vec{r}) = f(\vec{r}_0)$$

and it is linear, if

$$f(\vec{r}+\vec{s}) = f(\vec{r}) + f(\vec{s})$$

$$f(\lambda \vec{r}) = \lambda f(\vec{r}) \quad (\text{for arbitrary } \vec{r}, \vec{s}, \lambda)$$

A linear, vector function $f(\vec{r})$ is completely defined when $f(\vec{a_1})$, $f(\vec{a_2})$ and $f(\vec{a_3})$ are given for any three non-coplanar vectors $\vec{a_1}, \vec{a_2}, \vec{a_3}$. In terms of $\vec{a_1}, \vec{a_2}, \vec{a_3}$ as basis, we have,

$$\vec{r} = x_1 \vec{a_1} + x_2 \vec{a_2} + x_3 \vec{a_3}$$

$$\therefore f(\vec{r}) = f(x_1 \vec{a_1} + x_2 \vec{a_2} + x_3 \vec{a_3})$$

$$= f(x_1 \vec{a_1}) + f(x_2 \vec{a_2}) + f(x_3 \vec{a_3})$$

$$= x_1 f(\vec{a_1}) + x_2 f(\vec{a_2}) + x_3 f(\vec{a_3})$$

* Physical laws must be independent of any particular co-ordinate systems used in describing them mathematically if they are to be valid. A study of the consequences of this requirement leads to tensor analysis which has great use in general relativity theory, differential

geometry, mechanics, elasticity, hydrodynamics, electromagnetic theory and numerous other fields of science and engineering.

Spaces of N-dimensions :

In three dimensional space a point is a set of three numbers, called coordinates, determined by specifying a particular coordinate system or frame of reference. for example (x, y, z), (ρ, ϕ, z), (r, θ, ϕ) are coordinates of a point in rectangular, ~~spherical~~ cylindrical and spherical coordinate systems respectively. A point in N-dimensional space is, by analogy, a set of N numbers denoted by $(x^1, x^2, \ldots, x^N)$ where $1, 2, \ldots, N$ are taken not as exponents but as superscripts, a policy which will prove useful

The fact that we cannot visualize points in spaces of dimension higher than three has of course nothing whatsoever to do with their existence.

Coordinate transformations :

Let $(x^1, x^2, \ldots, x^N)$ and $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ be coordinates of a point in two different frames of reference. Suppose there exists N-independent

NB. In some books, $\bar{x}$ is written as x'.

relations betn the co-ordinates of the two systems having the form

$$\left.\begin{aligned} \bar{x}^1 &= \bar{x}^1(x^1, x^2, \ldots, x^N) \\ \bar{x}^2 &= \bar{x}^2(x^1, x^2, \ldots, x^N) \\ &\vdots \\ \bar{x}^N &= \bar{x}^N(x^1, x^2, \ldots, x^N) \end{aligned}\right\} \rightarrow (1)$$

which we can indicate briefly by

$$\bar{x}^K = \bar{x}^K(x^1, x^2, \ldots, x^N), \quad K = 1, 2, \ldots, N \rightarrow (2)$$

where it is supposed that the functions involved are single valued, continuous, and have continuous derivatives. Then conversely to each set of coordinates $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ there will correspond a unique set $(x^1, x^2, \ldots, x^N)$ given by

$$x^K = x^K(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N), \quad K = 1, 2, \ldots, N \rightarrow (3)$$

The relations (2) or (3) define a transformation of coordinates from one frame of reference to another.

The summation convention:

In writing an expression such as $a_1 x^1 + a_2 x^2 + \cdots + a_N x^N$, we can use the short notation $\sum_{j=1}^{N} a_j x^j$. An even shorter notation is simply to write it as $a_j x^j$, where we adopt the convention that whenever an index (subscript or superscript) is repeated in a given term we are to sum over that index from 1 to N unless otherwise specified. This is called the summation convention. Clearly, instead of using the index j we could have used another letter, say p, and the sum could be written $a_p x^p$. Any index which is repeated in a given term, so that the summation convention applies, is called a dummy index or umbral index

An index occurring only once in a given term is called a free index and can stand for any of the numbers 1, 2, ⋯, N such as K in eqn $\bar{x}^K = \bar{x}^K(x^1, x^2, \cdots, x^N)$, each of which represents N equations.

Contravariant and covariant Tensors :

If N quantities $A^1, A^2, \cdots, A^N$ in a coordinate system $(x^1, x^2, \cdots, x^N)$ are related to N other quantities $\bar{A}^1, \bar{A}^2, \cdots, \bar{A}^N$ in another coordinate system $(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^N)$ by the transformation equations

$$\bar{A}^p = \sum_{q=1}^{N} \frac{\partial \bar{x}^p}{\partial x^q} A^q, \quad p = 1, 2, \cdots, N$$

which by the conventions adopted can simply be written as

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^q} A^q$$

They are called components of a <u>contravariant vector</u> or contravariant tensor of the <u>1st rank</u> or <u>order</u>

If N quantities $A_1, A_2, \cdots, A_N$ in a coordinate system $(x^1, x^2, \cdots, x^N)$ are related to N other quantities $\bar{A}_1, \bar{A}_2, \cdots, \bar{A}_N$ in another co-ordinate system $(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^N)$ by the transformation equations

$$\bar{A}_p = \sum_{q=1}^{N} \frac{\partial x^q}{\partial \bar{x}^p} A_q \quad , \; p = 1, 2, \cdots, N$$

<u>or</u>

$$\bar{A}_p = \frac{\partial x^q}{\partial \bar{x}^p} A_q$$

They are called components of a <u>covariant vector</u> or <u>covariant tensor</u> of the <u>first rank</u> or <u>first order</u>

(Instead of speaking of a tensor whose components are A^p or A_p we shall often refer simply to the tensor A^p or A_p. No confusion should arise from this.)

<u>contravariant, covariant and mixed Tensors:</u>

If N^2 quantities A^{qs} in a coordinate system

$(x^1, x^2, \dots, x^N)$ are related to N^2 other quantities $\bar{A}^{pr}$ in another co-ordinate system $(\bar{x}^1, \bar{x}^2, \dots \bar{x}^N)$ by the transformation equations

$$\bar{A}^{pr} = \sum_{s=1}^{N} \sum_{q=1}^{N} \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^{qs}, \quad p, r = 1, 2, \dots, N$$

or

$$\bar{A}^{pr} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^{qs}$$

By the adopted conventions, they are called contravariant components of a tensor of the 2nd rank or rank two.

The N^2 quantities A_{qs} are called covariant components of a tensor of the 2nd rank if

$$\bar{A}_{pr} = \frac{\partial x^q}{\partial \bar{x}^p} \frac{\partial x^s}{\partial \bar{x}^r} A_{qs}$$

Similarly, the N^2 quantities A_s^q are called components of a mixed tensor of the second rank if

$$\bar{A}_r^p = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial x^s}{\partial \bar{x}^r} A_s^q$$

The Kronecker delta:

It is written by δ_k^j and is defined by

$$\delta_k^j = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } \cancel{j \neq k}\ j = k \end{cases}$$

As its notation indicates, it is a mixed tensor of the second rank.

Tensors of rank greater than two:

For example, A^{qst}_{kl} are the components of a mixed tensor of rank 5, contravariant of order 3 and covariant of order 2, if they transform according to the relations

$$\bar{A}^{prm}_{ij} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} \frac{\partial \bar{x}^m}{\partial x^t} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} A^{qst}_{kl}$$

Scalars or invariant:

Suppose ϕ is a f^n of the coordinates x^k, and let $\bar{\phi}$ denote the functional value under a transformation to a new set of coordinates $\bar{x}^k$. Then ϕ is called a scalar or invariant with respect to the co-ordinate transformation if $\phi = \bar{\phi}$. A scalar or invariant is also called a tensor of rank zero.

Tensor fields:

If to each point of a region in a N dimensional space, there corresponds a definite tensor, we say that a tensor field has been defined. This is a vector field or a scalar field according as the tensor is of rank one or zero. It should be noted that a tensor or tensor field is not just the set of ~~this~~ its components in one special coordinate system but all the possible sets under any transformation of coordinates.

Symmetric and Skew-symmetric tensors:
(or antisymmetric)

A tensor is called symmetric w.r.t. two contravariant or two covariant indices if its components remain unaltered upon interchange of the indices. Thus if $A^{mpr}_{qs} = A^{pmr}_{qs}$, the tensor is symmetric in m and p. If a tensor is symmetric w.r.t ~~any~~ two contravariant ~~and~~ or any two co-variant ~~component~~ indices, it is called symmetric.

A Tensor is called skew-symmetric w.r.t two contravariant or two covariant indices if its components change sign upon interchange of the indices. Thus if $A^{mpr}_{qs} = - A^{pmr}_{qs}$, the tensor is skew-symmetric in m and p. If a tensor is skew symmetric w.r.t. any two contravariant and any two covariant indices it is called skew-symmetric.

Fundamental operation with tensors :

Addition :

The sum of two or more tensors of the same rank and type (i.e. same no. of contravariant indices and same no. of covariant indices) is also a tensor of the same rank and type. Thus if A_q^{mp} and B_q^{mp} are tensors, then $C_q^{mp} = A_q^{mp} + B_q^{mp}$ is also a tensor. Addition of tensors is commutative and associative.

Subtraction :

The difference of two tensors of the same rank and type is also a tensor of the same rank and type. Thus if A_q^{mp} and B_q^{mp} are tensors, then $D_q^{mp} = A_q^{mp} - B_q^{mp}$ is also a tensor

Outer multiplication :

The product of two tensors is a tensor whose rank is the sum of the ranks of the given tensors. This product which involves ordinary multiplication of the components of the tensor is called the outer product. For example, $A_q^{pr} B_s^m = C_{qs}^{prm}$ is the outer product of A_q^{pr} and B_s^m. ~~However~~

note that ~~not~~ every tensor can not be written as a product of two tensors of lower rank. For this reason division of Tensors is not always possible.

Contraction :

If one contravariant and one covariant index of a tensor are set equal, the result indicates that a summation over the equal indices is to be taken according to the summation convention. This resulting sum is a tensor of rank two less than that of the original tensor. The process is called contraction. For example, ~~given the tensors A_q^{mp} and B_{st}^{r}~~ in the tensor of rank 5, A_{qs}^{mpr}, set $r=s$ to obtain $A_{qr}^{mpr} = B_q^{mp}$ a tensor of rank 3. Further by setting $p=q$, we obtain $B_p^{mp} = C^m$ a tensor of rank 1.

Inner Multiplication :

By the process of outer multiplication of two tensors followed by a contraction, we obtain a new tensor called an inner product of the given tensors. The process is called inner multiplication. For example, given the tensors A_q^{mp} and B_{st}^{r}, the outer product is $A_q^{mp} B_{st}^{r}$.

Letting $q = r$, we obtain the inner product $A^{mp}_{r} B^{r}_{st}$ $(= C^{mp}_{st})$.
Letting $q = r$ and $p = s$, another inner product $A^{mp}_{r} B^{r}_{pt}$ $(= C^{m}_{t})$ is obtained. Inner and outer multiplication of tensors is commutative and associative.

Quotient Law:

Suppose it is not known whether a quantity X is a Tensor or not. If an inner product of X with an arbitrary Tensor is itself a Tensor, Then X is also a Tensor. This is called The quotient law.