4. Derive Schroedinger's equation (time dependent). (6)

Ans. Schroedinger time dependent wave equation:

It is a differential eqn for the wave function $\psi(\vec{r},t)$ which represents the quantity varying periodically in a matter wave. The equation will have to be solved under specific boundary conditions and given forms of potential functions in special cases to determine $\psi(\vec{r},t)$ at any point of space and time

Although the potential function V may be a function of $(\vec{r},t)$, we consider the special case of a free particle for which $V=$ const. and construct the differential eqn for this special case. We assume that the same eqn is valid for a variable potential. So we have a situation of a free particle with constant $\lambda = \frac{h}{p}$ and $\nu = \frac{E}{h}$ such that the designed differential eqn will have a sinusoidally travelling wave solution with constant frequency and wave length.

Let $\psi(\vec{r},t) = A e^{i(\vec{K}\cdot\vec{r} - \omega t)}$

Now de Broglie eqn is

$$p = \frac{h}{\lambda} = \frac{2\pi h}{2\pi\lambda} = \hbar K$$

$$\therefore \vec{p} = \hbar\vec{K} \Rightarrow \vec{K} = \frac{\vec{p}}{\hbar}$$

Also $E = h\nu = \frac{2\pi h\nu}{2\pi} = \hbar\omega$

$$\Rightarrow \omega = \frac{E}{\hbar}$$

$$\therefore \psi(\vec{r},t) = A e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{r} - Et)}$$

$$= A e^{\frac{i}{\hbar}(p_x x + p_y y + p_z z - Et)}$$

$$\therefore \frac{\partial \psi}{\partial t} = A e^{\frac{i}{\hbar}(\vec{p}.\vec{r} - Et)} \cdot \frac{i}{\hbar}(-E)$$

$$= -\frac{iE}{\hbar}\psi = \frac{E}{i\hbar}\psi$$

$$\Rightarrow i\hbar \frac{\partial \psi}{\partial t} = E\psi \longrightarrow (1)$$

Again, $\frac{\partial \psi}{\partial x} = A e^{\frac{i}{\hbar}(p_x x + p_y y + p_z z - Et)} \cdot \frac{i}{\hbar} p_x$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = A e^{\frac{i}{\hbar}(p_x x + p_y y + p_z z - Et)} \cdot \frac{i}{\hbar} p_x \left(\frac{i}{\hbar} p_x\right)$$

$$= -\frac{p_x^2}{\hbar^2}\psi$$

$$\therefore \nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}$$

$$= -\frac{p_x^2 \psi}{\hbar^2} - \frac{p_y^2 \psi}{\hbar^2} - \frac{p_z^2 \psi}{\hbar^2}$$

$$= -\frac{\psi}{\hbar^2}(p_x^2 + p_y^2 + p_z^2)$$

$$= -\frac{\psi}{\hbar^2} p^2$$

$$\Rightarrow -\frac{\hbar^2}{2m}\nabla^2 \psi = \frac{\psi p^2}{2m} \longrightarrow (2)$$

We assume the classical expression for the total energy.

$$E = K.E. + P.E. = \frac{p^2}{2m} + V$$

$$\therefore (1) \Rightarrow i\hbar \frac{\partial \psi}{\partial t} = \left(\frac{p^2}{2m} + V\right)\psi = \frac{p^2 \psi}{2m} + V\psi$$

$$= -\frac{\hbar^2}{2m}\nabla^2 \psi + V\psi \quad (\text{using } (2))$$

$$\Rightarrow \boxed{-\frac{\hbar^2}{2m}\nabla^2 \psi + V\psi = i\hbar \frac{\partial \psi}{\partial t}}$$ This is the time dependent Schroedinger equation.